Core 4 Calculus Questions

2 A curve is defined by the parametric equations

$$x = 3 - 4t \qquad y = 1 + \frac{2}{t}$$

(a) Find $\frac{dy}{dx}$ in terms of t. *(4 marks)*

(b) Find the equation of the tangent to the curve at the point where $t = 2$, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers. *(4 marks)*

(c) Verify that the cartesian equation of the curve can be written as

$$(x - 3)(y - 1) + 8 = 0$$

(3 marks)

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6 (a) Express $\cos 2x$ in the form $a\cos^2 x + b$, where a and b are constants. *(2 marks)*

(b) Hence show that $\int_0^{\frac{\pi}{2}} \cos^2 x \, dx = \frac{\pi}{a}$, where a is an integer. *(5 marks)*

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8 (a) Solve the differential equation

$$\frac{dx}{dt} = -2(x - 6)^{\frac{1}{2}}$$

to find t in terms of x, given that $x = 70$ when $t = 0$. *(6 marks)*

(b) Liquid fuel is stored in a tank. At time t minutes, the depth of fuel in the tank is x cm. Initially there is a depth of 70 cm of fuel in the tank. There is a tap 6 cm above the bottom of the tank. The flow of fuel out of the tank is modelled by the differential equation

$$\frac{dx}{dt} = -2(x - 6)^{\frac{1}{2}}$$

(i) Explain what happens when $x = 6$. *(1 mark)*

(ii) Find how long it will take for the depth of fuel to fall from 70 cm to 22 cm. *(2 marks)*

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5 A curve is defined by the equation

$$y^2 - xy + 3x^2 - 5 = 0$$

(a) Find the y-coordinates of the two points on the curve where $x = 1$. *(3 marks)*

(b) (i) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y - 6x}{2y - x}$. *(6 marks)*

(ii) Find the gradient of the curve at each of the points where $x = 1$. *(2 marks)*

(iii) Show that, at the two stationary points on the curve, $33x^2 - 5 = 0$. *(3 marks)*

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7 Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 6xy^2$$

given that $y = 1$ when $x = 2$. Give your answer in the form $y = \mathrm{f}(x)$. *(6 marks)*

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1 A curve is defined by the parametric equations

$$x = 1 + 2t, \quad y = 1 - 4t^2$$

(a) (i) Find $\frac{\mathrm{d}x}{\mathrm{d}t}$ and $\frac{\mathrm{d}y}{\mathrm{d}t}$. *(2 marks)*

(ii) Hence find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t. *(2 marks)*

(b) Find an equation of the normal to the curve at the point where $t = 1$. *(4 marks)*

(c) Find a cartesian equation of the curve. *(3 marks)*

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8 (a) (i) Solve the differential equation $\frac{\mathrm{d}y}{\mathrm{d}t} = y \sin t$ to obtain y in terms of t. *(4 marks)*

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5 The point $P(1, a)$, where $a > 0$, lies on the curve $y + 4x = 5x^2y^2$.

(a) Show that $a = 1$. *(2 marks)*

(b) Find the gradient of the curve at P. *(7 marks)*

(c) Find an equation of the tangent to the curve at P. *(1 mark)*

6 A curve is given by the parametric equations

$$x = \cos\theta \qquad y = \sin 2\theta$$

(a) (i) Find $\dfrac{dx}{d\theta}$ and $\dfrac{dy}{d\theta}$. *(2 marks)*

(ii) Find the gradient of the curve at the point where $\theta = \dfrac{\pi}{6}$. *(2 marks)*

(b) Show that the cartesian equation of the curve can be written as

$$y^2 = kx^2(1 - x^2)$$

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8 (a) Solve the differential equation

$$\frac{dy}{dx} = \frac{\sqrt{1+2y}}{x^2}$$

given that $y = 4$ when $x = 1$. *(6 marks)*

(b) Show that the solution can be written as $y = \dfrac{1}{2}\left(15 - \dfrac{8}{x} + \dfrac{1}{x^2}\right)$. *(2 marks)*

Core 4 Calculus Answers

2(a)	$\frac{dy}{dt}=\frac{-2}{t^2}$ $\quad \frac{dx}{dt}=-4$	M1A1		
	$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{1}{\frac{dx}{dt}}=\frac{1}{2t^2}$	m1		Use chain rule
		A1F	4	Follow on use of chain rule $\left(\text{if f }(t)\right)$
				Or eliminate t: M1 $y=f(x)$ attempt to differentiate M1A1 chain rule A1F reintroduce t
(b)	$t=2 \quad m_T=\frac{1}{8}$	B1F		follow on gradient (possibly used later)
	$x=-5 \quad y=2$	B1		
	$y-2=\frac{1}{8}(x+5)$	M1		Their (x, y), m
	$x-8y+21=0$	A1F	4	Ft on (x, y) and m
(c)	$x-3=-4t \quad y-1=\frac{2}{t}$	M1		PI
	$(x-3)(y-1)=-4t\times\frac{2}{t}=(-8)$	M1		Attempt to eliminate t
		A1	3	AG convincingly obtained
	Total		**11**	

6(a)	$\cos 2x = 2\cos^2 x - 1$	B1B1	2	
(b)	$\cos^2 x = \frac{1}{2}(\cos 2x + 1)$	M1		Attempt to express $\cos^2 x$ in terms of $\cos 2x$
	$\frac{1}{2}\int_0^{\frac{\pi}{2}} \cos 2x + 1 \, dx = \left[\frac{1}{4}\sin 2x + \frac{x}{2}\right]_0^{\frac{\pi}{2}}$	A1 A1		
	$=\frac{\pi}{4}$	M1A1F	5	Use limits. Ft on integer *a*.
	Total		**7**	

8(a)	$\int \frac{\mathrm{d}x}{\sqrt{x-6}}\mathrm{d}x = \int -2\mathrm{d}t$	M1		Attempt to separate and integrate
	$2\sqrt{x-6} = -2t + c$	A1A1		c on either side
	$t = 0 \quad x=70 \quad \Rightarrow \quad c = 16$	m1A1F		Follow on c from sensible attempt at integrals $\left(\sqrt{\ } \text{ not ln}\right)$
	$t = 8 - \sqrt{x-6}$	A1	6	CAO (or AEF)
(b)(i)	The liquid level stops falling/flowing/ at minimum depth	B1	1	
	$x = 22 \qquad t = 8 - \sqrt{22-6}$	M1		Use $x = 22$ in their equation provided there is a c **Or** start again using limits M1 $2\sqrt{64} - 2\sqrt{16} = \pm 2t$, A1 $t = 4$
	$t = 4$	A1	2	CAO
	Total		**9**	

5(a)	$x = 1 \qquad y^2 - y + 3 - 5 = 0$	M1		
	$(y-2)(y+1) = 0$	M1		Attempt to solve quadratic equation with $x = 1$
	$y = 2 \qquad y = -1$	A1	3	
(b)(i)	$2y\frac{\mathrm{d}y}{\mathrm{d}x} - x\frac{\mathrm{d}y}{\mathrm{d}x} - y + 6x = 0$	B1B1 B1 M1A1		$+6x$; $-5 \to 0$ Chain rule Product rule (M1 two terms)
	$6x - y + (2y - x)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$	A1	6	Factorise and obtain answer given
	Alternative			
	$\frac{\mathrm{d}y}{\mathrm{d}x}(y-x)^2 = (y-x)(0-6x)$ $-\left(5-3x^2\right)\left(\frac{\mathrm{d}y}{\mathrm{d}x} - 1\right)$	(B1) (B1) (M1) (A1)		$5 \to 0$ $-6x$ Recognisable attempt at quotient rule Completely correct OE
	$\frac{\mathrm{d}y}{\mathrm{d}x}\left[(y+x)^2 + \left(5-3x^2\right)\right] = (y-x)(-6x)$ $+\left(5-3x^2\right)$	(A1)		Factorise out $\frac{\mathrm{d}y}{\mathrm{d}x}$
	Given answer	(A1)		Correct answer from correct working Be convinced
(ii)	$(1, 2) \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{4}{3}$	M1		Substitute $x = 1$ and one y value from (a)
	$(1, -1) \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{7}{3}$	A1F	2	Both; follow on candidates y s OE $\frac{-7}{-3}$; 3SF
(iii)	$y - 6x = 0$	B1		
	$(6x)^2 - x \times 6x + 3x^2 - 5 = 0$	M1		
	$36x^2 - 6x^2 + 3x^2 - 5 = 0$			
	$33x^2 - 5 = 0$	A1	3	AG convincingly obtained
	Total		**14**	

7	$\int \frac{dy}{y^2} = \int 6x \, dx$	M1		Attempt to separate Either d*x* or d*y* in right place
	$-\frac{1}{y} = 3x^2 (+C)$	A1A1		$-\frac{1}{y}$; $3x^2$
	$x = 2 \quad y = 1 \quad C = -13$	M1 A1		Use (2,1) to find a constant. CAO
	$y = \frac{1}{13 - 3x^2}$	A1	6	CAO OE
	Total		**6**	

1(a)(i)	$\frac{dx}{dt} = 2$, $\frac{dy}{dt} = -8t$	B1, B1	2	CAO
(ii)	$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{-8t}{2} = -4t$	M1 A1F	2	Chain rule in correct form ft on sign coefficient errors (**not** power of *t*)
(b)	$m_T = -4$, $m_N = \frac{1}{4}$	B1F, B1F		ft on $\frac{dy}{dx}$ if f(*t*)
	$x = 3 \quad y = -3$			
	$\frac{y - -3}{x - 3} = \frac{1}{4} \Rightarrow \frac{y + 3}{x - 3} = \frac{1}{4}$	M1 A1	4	Use candidate's (x, y) and m_N Any correct form; ISW; CAO
(c)	$t = \frac{x - 1}{2}$	M1		
	$y = 1 - 4\left(\frac{x - 1}{2}\right)^2$	M1A1	3	Substitute for *t* Simplification not required but CAO Or equivalent methods / forms: $y = 2x - x^2$, $t^2 = \frac{1 - y}{4}$, $\left(\frac{x - 1}{2}\right)^2 = \frac{1 - y}{4}$
	Total		**11**	

8(a)(i)	$\int \frac{dy}{y} = \int \sin t \, dt$	M1		Attempt to separate and integrate
	$\ln y = -\cos t + C$	A1,A1		A1 for ln *y*; A1 for $-\cos t$; condone missing *C*
	$y = Ae^{-\cos t}$	A1	4	*A* present; or $y = e^{-\cos t + C}$

Q	Solution	Marks	Total	Comments
5(a)	$x=1,\ 5a^2-a-4=0$	M1		condone y for a
	$(5a+4)(a-1)=0,\quad a=1$	A1	2	AG – be convinced, both factors seen or $a=-\frac{4}{5}$ or $1 \Rightarrow a=1$ A0 for 2 positive roots (substitute $(1,1) \Rightarrow 5=5$ no marks)
(b)	$\frac{dy}{dx}+4$	B1B1		(Ignore '$\frac{dy}{dx}=$' if not used, otherwise loses final A1)
	$=10xy^2+10x^2y\frac{dy}{dx}$	M1		attempt product rule, see two terms added
		M1		chain rule, $\frac{dy}{dx}$ attached to one term only
		A1		condone 5×2 for 10
	$x=1, y=1\quad \frac{dy}{dx}+4=10+10\frac{dy}{dx}$	M1		two terms, or more, in $\frac{dy}{dx}$
	$\frac{dy}{dx}=-\frac{6}{9}=\left(-\frac{2}{3}\right)$	A1	7	CSO
	Alt (for last two marks)			
	$\frac{dy}{dx}=\frac{10xy^2-4}{1-10x^2y}$	(M1)		find $\frac{dy}{dx}$ in terms of x, y and substitute $x=1,\ y=1$ must be from expression with two terms or more in $\frac{dy}{dx}$
	$(1,1) \Rightarrow \frac{10-4}{1-10}=-\frac{6}{9}$	(A1)		
(c)	$\frac{y-1}{x-1}=-\frac{2}{3}$ (OE)	B1F	1	ft on gradient ISW after any correct form
	Total		**10**	

6(a)(i)	$\frac{dx}{d\theta} = -\sin\theta \quad \frac{dy}{d\theta} = 2\cos 2\theta$	B1 B1	2	
(ii)	$\frac{dy}{dx} = -\frac{2\cos 2\theta}{\sin\theta}, \ \frac{dy}{dx} = -\frac{2\cos\frac{\pi}{3}}{\sin\frac{\pi}{6}} = -2$	M1		use chain rule their $\frac{dy}{d\theta}$ / their $\frac{dx}{d\theta}$ and
				substitute $\theta = \frac{\pi}{6}$
		A1	2	
(b)	$y = 2\sin\theta\cos\theta = 2\sqrt{1-\cos^2\theta}\cos\theta$	B1		use $\sin 2\theta = 2\sin\theta\cos\theta$
		B1		use $\sin^2\theta = 1-\cos^2\theta$
	$y = 2\sqrt{1-x^2}\ x$	M1		$\sin\theta$, $\cos\theta$ in terms of x
	$y^2 = 4x^2(1-x^2)$	A1	4	all correct CSO
	Alt			
	$y^2 = \sin^2 2\theta = (2\sin\theta\cos\theta)^2$	(B1)		use of double angle formula
	$= (4)\sin^2\theta\cos^2\theta = (4)(1-\cos^2\theta)\cos^2\theta$	(B1)		use of $s^2 + c^2 = 1$ to eliminate $\sin\theta$
	$= (4)(1-x^2)x^2$	(M1)		Substitute $\cos\theta$ for x
	$= 4(1-x^2)x^2$	(A1)	(4)	CSO
	Total		**8**	

8(a)	$\int \frac{1}{\sqrt{1+2y}}\,dy = \int \frac{1}{x^2}\,dx$	M1		attempt to separate and integrate
	$\int \frac{1}{\sqrt{1+2y}}\,dy = k\sqrt{1+2y}$	m1		
		A1		OE A1 for $\sqrt{1+2y}$ depends on both Ms
	$\sqrt{1+2y} = -\frac{1}{x}(+c)$	A1		A1 for $-\frac{1}{x}$ depends on first M1 only
	$x = 1, y = 4 \Rightarrow c = 4$	m1		$+c$ must be seen on previous line
		A1F	6	ft on k and $\pm\frac{1}{x}$ only
(b)	$1+2y = \left(4-\frac{1}{x}\right)^2$	m1		need $k\sqrt{1+2y}$ = 'x expression with $+ c$' **and** attempt to square both sides
	$2y = 15 + \frac{1}{x^2} - \frac{8}{x}$	A1	2	terms on RHS in any order AG – be convinced CSO
	Total		**8**	