FP1 Mock Paper

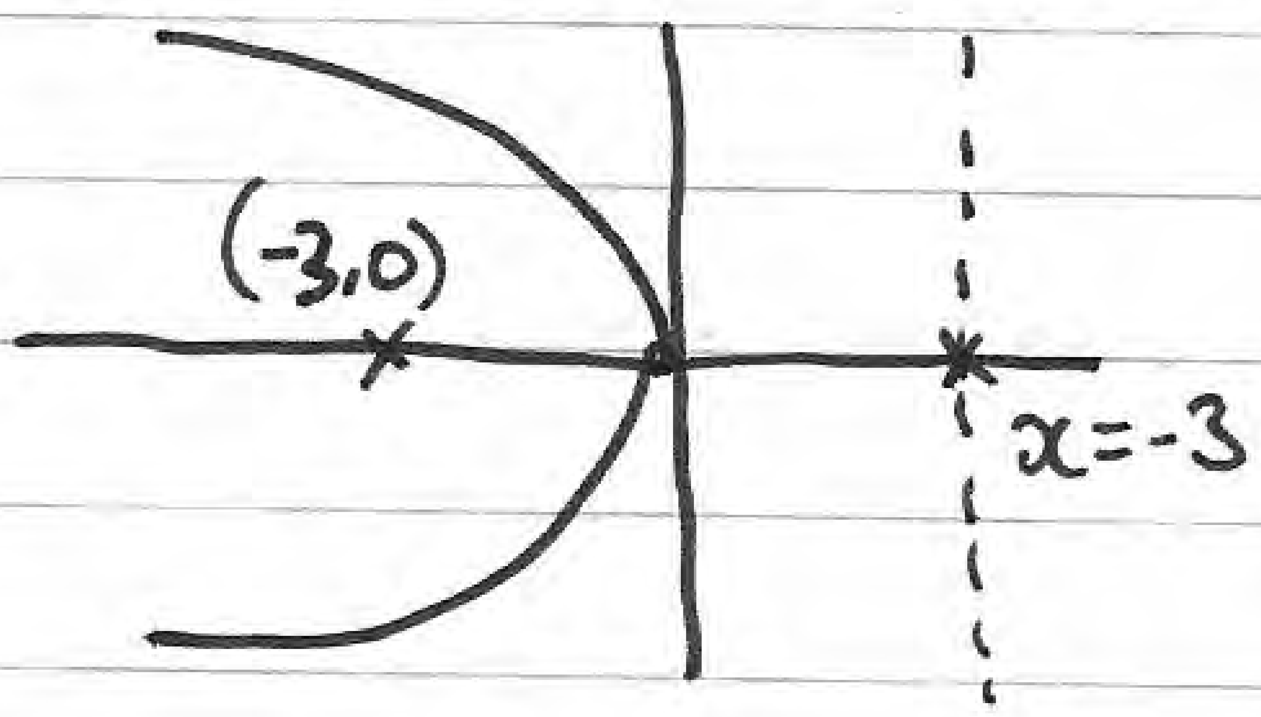
1) $R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $S = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

a) $R^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

b) $RS = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

rotation 180° about origin

2)



$y^2 = 4ax$

$y^2 = -12x$

3) $z = 1 + i\sqrt{3}$

$z^2 = (1+i\sqrt{3})(1+i\sqrt{3})$

$= 1 + 3i^2 + 2\sqrt{3}\,i$

$= -2 + i2\sqrt{3}$

$z + z^2 = 1 + i\sqrt{3}$

$\quad -2 + i2\sqrt{3}$

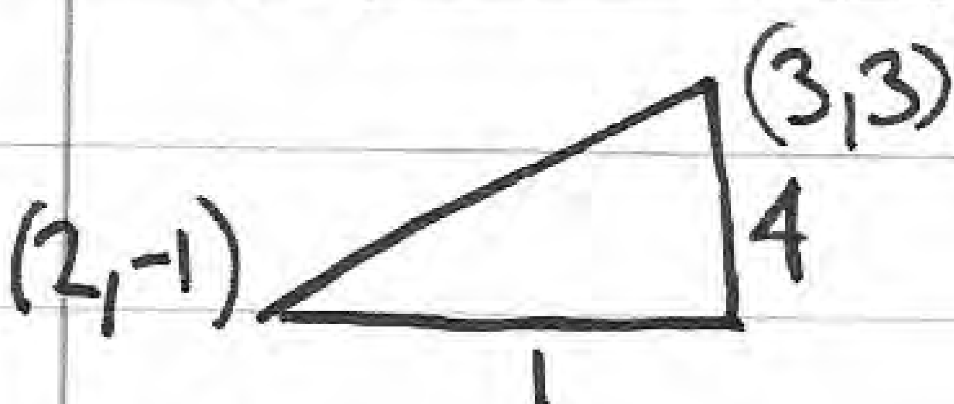
$\underline{-1 + i3\sqrt{3}}$

b) $\dfrac{z}{3-z} = \dfrac{1+i\sqrt{3}}{3-1-i\sqrt{3}} = \dfrac{1+i\sqrt{3} \times (2+i\sqrt{3})}{2-i\sqrt{3} \times (2+i\sqrt{3})}$

$= \dfrac{2+3i^2+i3\sqrt{3}}{4-3i^2} = \dfrac{-1+i3\sqrt{3}}{7} = -\dfrac{1}{7} + i\dfrac{3\sqrt{3}}{7}$

4) $f(x) = x^3 - 4x^2 + 5x - 3$ $\quad f(2) = -1 \quad f(3) = 3$

$(2,-1) \rightarrow (3,3)$



$m = \dfrac{4}{1}$

$y + 1 = 4(x-2)$

$y = 0 \Rightarrow \frac{1}{4} = x - 2 \Rightarrow x = 2\frac{1}{4}$

b) $f'(x) = 3x^2 - 8x + 5$

$x_0 = 2.5 \qquad x_1 = 2.5 - \dfrac{f(2.5)}{f'(2.5)} = \dfrac{37}{15} = \underline{2.47 \text{ (2dp)}}$

5) $X = \begin{pmatrix} a & 2b \\ -a & 3b \end{pmatrix} \qquad \det X = 3ab + 2ab = 5ab$

$X^{-1} = \dfrac{1}{5ab}\begin{pmatrix} 3b & -2b \\ a & a \end{pmatrix}$

b) $ZX = Y \Rightarrow ZXX^{-1} = YX^{-1} \Rightarrow Z = YX^{-1}$

$Z = \begin{pmatrix} 3a & b \\ a & 2b \end{pmatrix}\begin{pmatrix} 3b & -2b \\ a & a \end{pmatrix}\dfrac{1}{5ab} = \dfrac{1}{5ab}\begin{pmatrix} 9ab + ab & -6ab + ab \\ 3ab + 2ab & -2ab + 2ab \end{pmatrix}$

$Z = \dfrac{1}{5ab}\begin{pmatrix} 10ab & -5ab \\ 5ab & 0 \end{pmatrix} \Rightarrow Z = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$

6) $\sum_1^n r(2r^2-6) = 2\sum_1^n r^3 - 6\sum_1^n r = 2\left(\frac{1}{4}n^2(n+1)^2\right) - 6\left(\frac{1}{2}n(n+1)\right)$

$= \frac{1}{2}n(n+1)\left[n(n+1) - 6\right] = \frac{1}{2}n(n+1)(n^2+n-6)$

$= \frac{1}{2}n(n+1)(n+3)(n-2)$

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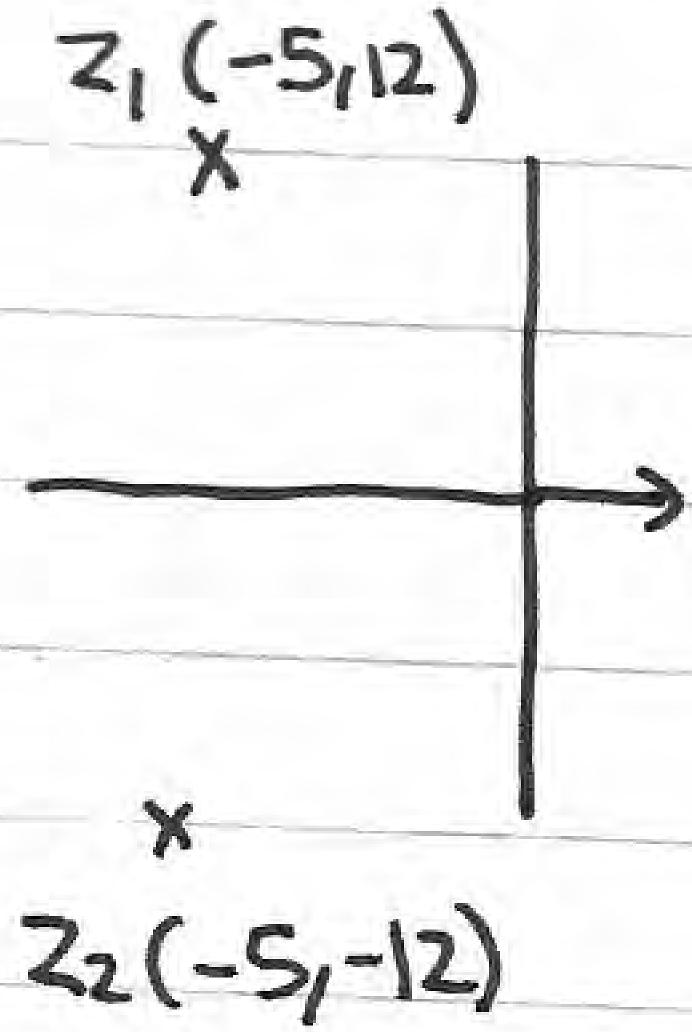
b) $\sum_{10}^{50} r(2r^2-6) = \frac{1}{2}(50)(51)(53)(48) - \frac{1}{2}(9)(10)(12)(7) = \underline{3239820}$

7) $z^2 + 10z + 169 = 0 \Rightarrow (z+5)^2 - 25 = -169 \Rightarrow (z+5)^2 = -144$

$\Rightarrow z = -5 \pm \sqrt{-144} \Rightarrow z_1 = \underline{-5 + 12i}, \; z_2 = \underline{-5 - 12i}$

b) $|z_1| = |z_2| = \sqrt{5^2 + 12^2} = 13$

c)



$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{12}{-5} \Rightarrow \theta = \tan^{-1}\left(\frac{12}{-5}\right)$

$\theta = -1.176$

$+\pi$

$\arg(z_1) = 1.97$

$\Rightarrow \arg(z_2) = -1.97$

d $|z_1 - z_2| = |-5+12i - (-5-12i)| = |24i| = \underline{24}$

8) $xy = c^2$ $\quad (3t, \frac{3}{t}) \Rightarrow 3t\left(\frac{3}{t}\right) = c^2 \Rightarrow \underline{9 = c^2}$ $\quad c = \pm 3$

b) $xy = 9$ $\quad y = 9x^{-1} \Rightarrow \frac{dy}{dx} = -9x^{-2} = \frac{-9}{x^2} = \frac{-9}{9t^2}$

$m_t = \frac{-1}{t^2} \Rightarrow m_n = t^2$ $\quad y - \frac{3}{t} = t^2(x - 3t)$

$\Rightarrow y - \frac{3}{t} = t^2x - 3t^3$ $\quad \therefore y = t^2x + \left(\frac{3}{t} - 3t^3\right)$

c) $H(6, 1.5) = \left(3t, \frac{3}{t}\right) \Rightarrow t = 2$ $\Rightarrow y = 4x + \left(\frac{3}{2} - 3(2)^3\right)$

$y = 4x - \frac{45}{2}$

$xy = 9 \Rightarrow x\left(4x - \frac{45}{2}\right) = 9 \Rightarrow 4x^2 - \frac{45}{2}x - 9 = 0$

$\Rightarrow 8x^2 - 45x - 18 = 0$ $(x-6)(8x+3) = 0$

$x = 6 \checkmark$ $x = -\frac{3}{8}$ $y = -\frac{12}{8} - \frac{45}{2} = -24$ $Q\left(-\frac{3}{8}, -24\right)$

a) $u_1 = 3 \quad u_{n+1} = 3u_n + 4 \qquad \text{prove } u_n = 3^n + 2(3^{n-1} - 1)$

$n=1 \quad u_1 = 3 \quad u_1 = 3^1 + 2(3^{1-1} - 1) = 3^1 = 3 \quad \checkmark$

$n=2 \quad u_2 = 3(3) + 4 = 13 \quad u_2 = 3^2 + 2(3^{2-1} - 1) = 9 + 4 = 13 \checkmark$

$n = k+1 \quad u_{k+1} = 3u_k + 4 = 3\left(3^k + 2(3^{k-1} - 1)\right) + 4$

$\Rightarrow u_{k+1} = 3^{k+1} + 2[3(3^{k-1} - 1)] + 4$

$\Rightarrow u_{k+1} = 3^{k+1} + 2(3^k - 3) + 2 \times 2$

$\Rightarrow u_{k+1} = 3^{k+1} + 2(3^k - 3 + 2)$

$\therefore u_{k+1} = 3^{k+1} + 2(3^k - 1)$

true for $n=1, n=2$; true for $n=k+1$, if true for $n=k$
$\therefore$ by Induction, true for all $n \in \mathbb{Z}^+$.

b) $A = \begin{pmatrix} 4 & 0 \\ 9 & 1 \end{pmatrix} \qquad \text{Prove } A^n = \begin{pmatrix} 4^n & 0 \\ 3(4^n - 1) & 1 \end{pmatrix}$

$n=1 \quad A^1 = A = \begin{pmatrix} 4^1 & 0 \\ 3(4^1 - 1) & 1 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 9 & 1 \end{pmatrix} \quad \checkmark$

$n = k+1 \Rightarrow A^{k+1} = \begin{pmatrix} 4^{k+1} & 0 \\ 3(4^{k+1} - 1) & 1 \end{pmatrix}$

$A^{k+1} = AA^k = \begin{pmatrix} 4 & 0 \\ 9 & 1 \end{pmatrix} \begin{pmatrix} 4^k & 0 \\ 3(4^k - 1) & 1 \end{pmatrix} = \begin{pmatrix} 4 \times 4^k & 0 \\ 9 \times 4^k + 3(4^k - 1) & 1 \end{pmatrix}$

$9(4^k) + 3(4^k - 1) = 12(4^k) - 3 = 3 \times 4(4^k) - 3 = 3(4^{k+1} - 1)$

true for $n=1$, true for $n=k+1$ if true for $n=k$
$\therefore$ by Induction true for all $n \in \mathbb{Z}^+$

(ii) $A^{-1}\begin{pmatrix} 4^{-1} & 0 \\ 3(4^{-1}-1) & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & 0 \\ 3(\frac{1}{4}-1) & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & 0 \\ \frac{-9}{4} & 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 & 0 \\ -9 & 4 \end{pmatrix}$

$\det A = 4 - 0 = 4 \quad A^{-1} = \frac{1}{\det A}\begin{pmatrix} 1 & 0 \\ -9 & 4 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 & 0 \\ -9 & 4 \end{pmatrix}$ ✓

$\therefore$ It is also valid for $n = -1$