Cambridge Assessment International Education

Cambridge International AS & A Level

CANDIDATE NAME												
CENTRE NUMBER							CANDIDATE NUMBER					

2388089987

FURTHER MATHEMATICS **9231/13**

Paper 1 Further Pure Mathematics 1 **May/June 2020**

2 hours

You must answer on the question paper.

You will need: List of formulae (MF19)

INSTRUCTIONS

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

INFORMATION

- The total mark for this paper is 75.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **16** pages. Blank pages are indicated.

1 The cubic equation $7x^3 + 3x^2 + 5x + 1 = 0$ has roots α, β, γ.

(a) Find a cubic equation whose roots are α^{-1}, β^{-1}, γ^{-1}. [3]

(b) Find the value of $\alpha^{-2} + \beta^{-2} + \gamma^{-2}$. [2]

(c) Find the value of $\alpha^{-3} + \beta^{-3} + \gamma^{-3}$. [2]

2 The sequence $u_1, u_2, u_3, \ldots$ is such that $u_1 = 1$ and $u_{n+1} = 2u_n + 1$ for $n \geqslant 1$.

(a) Prove by induction that $u_n = 2^n - 1$ for all positive integers n. [5]

$u_1 = 1 = 2^1 - 1$

Assume that it is true for $n=k$

So $u_k = 2^k - 1$

Then $u_{k+1} = 2(2^k - 1) + 1$

$= 2^{k+1} - 1$

So, it is also true for $n=k+1$

Hence, by induction, true for all posit integers

(b) Deduce that u_{2n} is divisible by u_n for $n \geqslant 1$. [2]

$a^2 - b^2$

$$\frac{u_{2n}}{u_n} = \frac{2^{2n}-1}{2^n-1} = \frac{(2^n-1)(2^n+1)}{2^n-1}$$

$= 2^n + 1$

3 Let $S_n = 2^2 + 6^2 + 10^2 + \ldots + (4n-2)^2$.

(a) Use standard results from the List of Formulae (MF19) to show that $S_n = \frac{4}{3}n(4n^2-1)$. [4]

(b) Express $\frac{n}{S_n}$ in partial fractions and find $\sum_{n=1}^{N} \frac{n}{S_n}$ in terms of N. integer [4]

$$\frac{n}{S_n} = \frac{3}{4(2n-1)(2n+1)} = \frac{3}{8}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$$

$$\frac{3 \cdot 1}{4(2n-1)(2n+1)} = \left(\frac{A}{2n-1} + \frac{B}{2n+1}\right)\frac{3}{4}$$

$$A(2n+1) + B(2n-1)$$

A :

B :

$$\sum_{n=1}^{N} \frac{n}{S_n} = \frac{3}{8}\left(1 - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{5}} + \cancel{\frac{1}{5}} - \cancel{\frac{1}{7}} \cancel{+} \cancel{\frac{1}{2N-1}} - \frac{1}{2N+1}\right)$$

$$= \frac{3}{8}\left(1 - \frac{1}{2N+1}\right)$$

(c) Deduce the value of $\sum_{n=1}^{\infty} \frac{n}{S_n}$. [1]

$$\sum_{n=1}^{\infty} \frac{n}{S_n} = \frac{3}{8}$$

4 The matrix **A** is given by

$$\mathbf{A} = \begin{pmatrix} k & 0 & 2 \\ 0 & -1 & -1 \\ 1 & 1 & -k \end{pmatrix},$$

where k is a real constant.

(a) Show that **A** is non-singular. [3]

$\det(A) = k\begin{vmatrix} -1 & -1 \\ 1 & -k \end{vmatrix} - 0 + 2\begin{vmatrix} 0 & -1 \\ 1 & 1 \end{vmatrix}$

$= k(k+1) + 2\cdot 1$

$= k^2 + k + 2 \neq 0$

no $\quad k^2 - 1 \quad (k+\tfrac{1}{2})^2 + 1\tfrac{3}{4}$

The matrices **B** and **C** are given by

$$\mathbf{B} = \begin{pmatrix} 0 & -3 \\ -1 & 3 \\ 0 & 0 \end{pmatrix} \text{ and } \mathbf{C} = \begin{pmatrix} -3 & -1 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

$CAB = C(AB) = (CA)B \neq CBA$

It is given that $\mathbf{CAB} = \begin{pmatrix} -2 & -\frac{3}{2} \\ -1 & -\frac{3}{2} \end{pmatrix}$.

(b) Find the value of k. [3]

$CAB = \begin{pmatrix} -3 & -1 & 1 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} k & 0 & 2 \\ 0 & -1 & -1 \\ 1 & 1 & -k \end{pmatrix}\begin{pmatrix} 0 & -3 \\ -1 & 3 \\ 0 & 0 \end{pmatrix}$

$= \begin{pmatrix} -3 & -1 & 1 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} 0 & -3k \\ 1 & -3 \\ -1 & 0 \end{pmatrix}$

$= \begin{pmatrix} -2 & 9k+3 \\ -1 & -3k-3 \end{pmatrix} = \begin{pmatrix} -2 & -\frac{3}{2} \\ -1 & -\frac{3}{2} \end{pmatrix}$

 $y = mx$

(c) Find the equations of the invariant lines, through the origin, of the transformation in the x-y plane represented by **CAB**. [5]

$$\begin{pmatrix} 2 & \frac{3}{2} \\ 1 & \frac{3}{2} \end{pmatrix} \begin{pmatrix} t \\ mt \end{pmatrix} = \begin{pmatrix} T \\ mT \end{pmatrix}$$

$$2t + \frac{3}{2}mt = T$$

$$t + \frac{3}{2}mt = mT$$

$$\frac{2 + \frac{3}{2}m}{1 + \frac{3}{2}m} = \frac{1}{m} \qquad \frac{1 + \frac{3}{2}}{2 + \frac{3}{2}m} = m$$

$$2m + \frac{3}{2}m^2 = 1 + \frac{3}{2}m$$

$m = -1$ $\quad m =$

$$\boxed{y = mx}$$

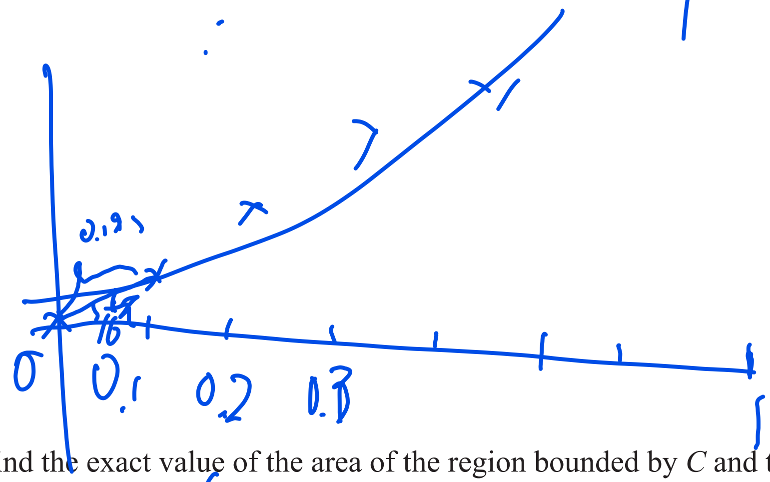
$\int r \, d\theta$

5 The curve C has polar equation $r = a\tan\theta$, where a is a positive constant and $0 \leqslant \theta \leqslant \frac{1}{4}\pi$.

(a) Sketch C and state the greatest distance of a point on C from the pole. [2]

$r = \tan\theta$

$\theta =$	0	$\frac{1}{16}\pi$	$\frac{1}{8}\pi$	$\frac{1}{4}\pi$
$r =$	0	0.198		



(b) Find the exact value of the area of the region bounded by C and the half-line $\theta = \frac{1}{4}\pi$. [4]

$\int r \, d\theta$

$$A = \frac{1}{2}\int_0^{\frac{1}{4}\pi} r^2 \, d\theta$$

(c) Show that C has Cartesian equation $y = \dfrac{x^2}{\sqrt{a^2 - x^2}}$. [3]

$\cos = \frac{x}{r}$ $r = a \tan\theta$

$\leftarrow x = r\cos\theta$ $x^2 + y^2 = r^2$

$\leftarrow y = r\sin\theta$

$\sin = \frac{y}{r}$ $\sqrt{x^2+y^2} = a\frac{\sin\theta}{\cos\theta}$

$= a\frac{\frac{y}{r}}{\frac{x}{r}} = a\frac{y}{x}$

(d) Using your answer to part **(b)**, deduce the exact value of $\displaystyle\int_0^{\frac{1}{2}a\sqrt{2}} \frac{x^2}{\sqrt{a^2-x^2}}\,\mathrm{d}x$. [2]

$\displaystyle\int_0^{\frac{1}{2}a\sqrt{2}} \frac{x^2}{\sqrt{a^2-x^2}}\,dx$ y $r = a\tan\theta$

$\frac{1}{2}a^2\displaystyle\int_0^{\frac{1}{4}} \tan^2\theta\, d\theta$

$= \dfrac{(\frac{1}{2}a\sqrt{2})^2}{\sqrt{a^2 - (\frac{1}{2}a\sqrt{2})^2}} - 0$

$= \dfrac{\frac{1}{2}a^2}{\sqrt{a^2 - \frac{1}{2}a^2}} = \dfrac{\frac{1}{2}a^2}{\sqrt{\frac{1}{2}a^2}} = \frac{a}{2}\sqrt{2}$

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6 The curve C has equation $y = \dfrac{10 + x - 2x^2}{2x - 3}$.

(a) Find the equations of the asymptotes of C. [3]

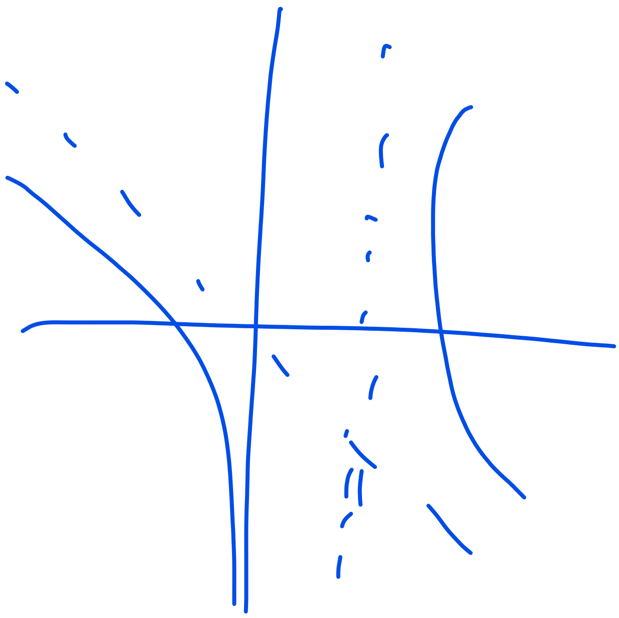
(b) Show that C has no turning points. [3]

$\frac{dy}{dx}$

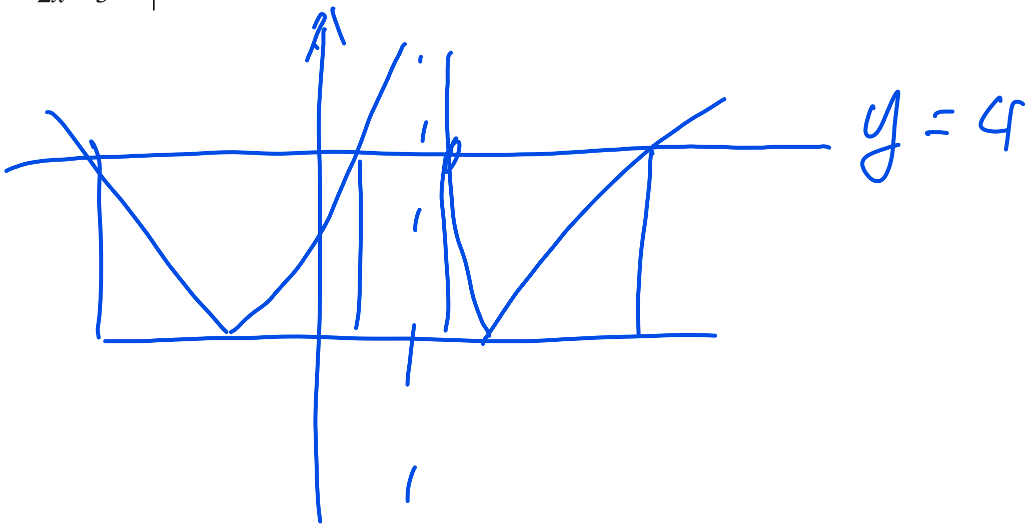
(c) Sketch C, stating the coordinates of the intersections with the axes. [3]

x - axis when $y = 0$ $x = ?$

y - axi $x = 0$ $y = ?$



(d) Sketch the curve with equation $y = \left|\dfrac{10+x-2x^2}{2x-3}\right|$ and find in exact form the set of values of x for which $\left|\dfrac{10+x-2x^2}{2x-3}\right| < 4$. [6]



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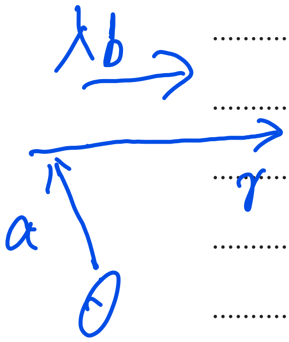
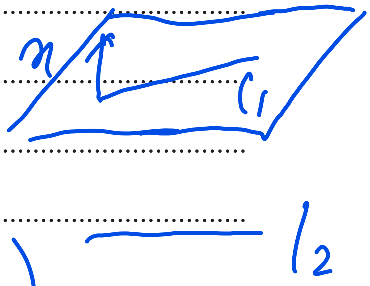
7 The lines l_1 and l_2 have equations $\mathbf{r} = -5\mathbf{j} + \lambda(5\mathbf{i} + 2\mathbf{k})$ and $\mathbf{r} = 4\mathbf{i} + 2\mathbf{j} - 2\mathbf{k} + \mu(\mathbf{j} + \mathbf{k})$ respectively. The plane Π contains l_1 and is parallel to l_2.

(a) Find the equation of Π, giving your answer in the form $ax + by + cz = d$. $= rn = an$ [4]



$l_1: \ r = \begin{pmatrix} 0 \\ -5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix}$



$l_2: \ r = \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix} + u \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$n = \begin{vmatrix} i & j & k \\ 5 & 0 & 2 \\ 0 & 1 & 1 \end{vmatrix} = \begin{pmatrix} -2 \\ -5 \\ 5 \end{pmatrix}$

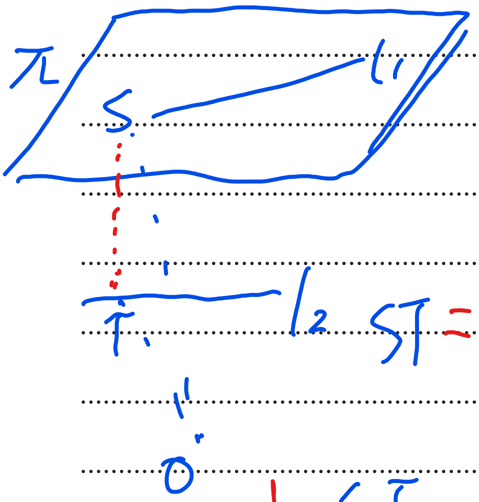
$d = a.n$

$\begin{pmatrix} 0 \\ -5 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ -5 \\ 5 \end{pmatrix} = 25$

(b) Find the distance between l_2 and Π. [3]

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -2 \\ -5 \\ 5 \end{pmatrix} = 25$

$\underline{-2x - 5y + 5z = 25}$

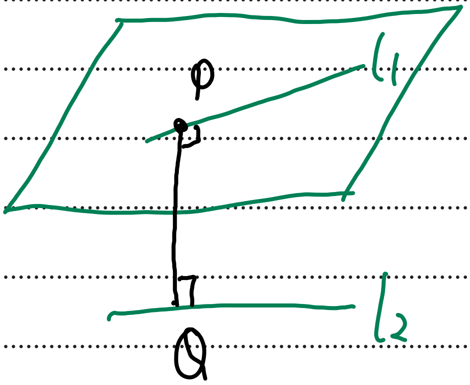


$ST = \begin{pmatrix} 0 \\ -5 \\ 0 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} -4 \\ -7 \\ 2 \end{pmatrix}$

$\left| \dfrac{ST \cdot n}{|n|} \right| = \left| \dfrac{\begin{pmatrix} -4 \\ -7 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ -5 \\ 5 \end{pmatrix}}{\left|\sqrt{(-2)^2 + (-5)^2 + 5^2}\right|} \right| = \dfrac{53}{\sqrt{54}}$

The point P on l_1 and the point Q on l_2 are such that PQ is perpendicular to both l_1 and l_2.

(c) Show that P has position vector $\frac{55}{27}\mathbf{i} - 5\mathbf{j} + \frac{22}{27}\mathbf{k}$ and state a vector equation for PQ. [8]



$OP = \begin{pmatrix} 5\lambda \\ -5 \\ 2\lambda \end{pmatrix}$

$OQ = \begin{pmatrix} 4 \\ 2+\mu \\ -2+\mu \end{pmatrix}$

$\overline{PQ} = OQ - OP = \begin{pmatrix} 4-5\lambda \\ 7+\mu \\ -2+\mu-2\lambda \end{pmatrix}$

$\begin{pmatrix} 4-5\lambda \\ 7+\mu \\ -2+\mu-2\lambda \end{pmatrix}\begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix} = 0$ ①

$-29\lambda + 2\mu = -16$

$\begin{pmatrix} \\ \\ \\ \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 0$ ②

$-2\lambda + 2\mu = -5$

$\lambda = \frac{11}{27}$

$\mu =$

$OP =$

Additional Page

If you use the following lined page to complete the answer(s) to any question(s), the question number(s) must be clearly shown.