UNIVERSITY OF CAMBRIDGE INTERNATIONAL EXAMINATIONS
General Certificate of Education
Advanced Level

FURTHER MATHEMATICS **9231/01**

Paper 1

October/November 2004

3 hours

Additional materials: Answer Booklet/Paper
Graph paper
List of Formulae (MF10)

READ THESE INSTRUCTIONS FIRST

If you have been given an Answer Booklet, follow the instructions on the front cover of the Booklet.
Write your Centre number, candidate number and name on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
At the end of the examination, fasten all your work securely together.

The number of marks is given in brackets [ ] at the end of each question or part question.
The use of a calculator is expected, where appropriate.
Results obtained solely from a graphic calculator, without supporting working or reasoning, will not receive credit.
You are reminded of the need for clear presentation in your answers.

This document consists of **5** printed pages and **3** blank pages.

 UNIVERSITY of CAMBRIDGE International Examinations

1 The linear transformation $\mathrm{T} : \mathbb{R}^4 \to \mathbb{R}^4$ is represented by the matrix

$$\begin{pmatrix} 1 & 5 & 2 & 6 \\ 2 & 0 & -1 & 7 \\ 3 & -1 & -2 & 10 \\ 4 & 10 & 13 & 29 \end{pmatrix}.$$

Find the dimension of the null space of T. [4]

2 The curve C is defined parametrically by

$$x = a\cos^3 t, \quad y = a\sin^3 t, \qquad 0 \leqslant t \leqslant \tfrac{1}{2}\pi,$$

where a is a positive constant. Find the area of the surface generated when C is rotated through one complete revolution about the x-axis. [5]

3 Given that

$$\alpha + \beta + \gamma = 0, \quad \alpha^2 + \beta^2 + \gamma^2 = 14, \quad \alpha^3 + \beta^3 + \gamma^3 = -18,$$

find a cubic equation whose roots are α, β, γ. [4]

Hence find possible values for α, β, γ. [2]

4 The curve C has polar equation

$$r = \mathrm{e}^{\frac{1}{5}\theta}, \qquad 0 \leqslant \theta \leqslant \tfrac{3}{2}\pi.$$

(i) Draw a sketch of C. [2]

(ii) Find the length of C, correct to 3 significant figures. [4]

5 Let

$$S_N = \sum_{n=1}^{N} (-1)^{n-1} n^3.$$

Find S_{2N} in terms of N, simplifying your answer as far as possible. [4]

Hence write down an expression for S_{2N+1} and find the limit, as $N \to \infty$, of $\dfrac{S_{2N+1}}{N^3}$. [3]

6 Write down all the 8th roots of unity. [2]

Verify that

$$\left(z - \mathrm{e}^{\mathrm{i}\theta}\right)\left(z - \mathrm{e}^{-\mathrm{i}\theta}\right) \equiv z^2 - (2\cos\theta)z + 1. \quad [1]$$

Hence express $z^8 - 1$ as the product of two linear factors and three quadratic factors, where all coefficients are real and expressed in a non-trigonometric form. [5]

7 The curve C has equation

$$xy + (x + y)^5 = 1.$$

(i) Show that $\frac{dy}{dx} = -\frac{5}{6}$ at the point $A\ (1, 0)$ on C. [3]

(ii) Find the value of $\frac{d^2y}{dx^2}$ at A. [5]

8 The sequence of real numbers $a_1, a_2, a_3, \ldots$ is such that $a_1 = 1$ and

$$a_{n+1} = \left(a_n + \frac{1}{a_n}\right)^{\lambda},$$

where λ is a constant greater than 1. Prove by mathematical induction that, for $n \geqslant 2$,

$$a_n \geqslant 2^{g(n)},$$

where $g(n) = \lambda^{n-1}$. [6]

Prove also that, for $n \geqslant 2$, $\frac{a_{n+1}}{a_n} > 2^{(\lambda-1)g(n)}$. [3]

9 It is given that

$$I_n = \int_0^1 (1 + x^3)^{-n}\, dx,$$

where $n > 0$.

(i) Show that

$$\frac{d}{dx}\left[x(1 + x^3)^{-n}\right] = -(3n - 1)(1 + x^3)^{-n} + 3n(1 + x^3)^{-n-1},$$

and hence, or otherwise, show that

$$I_{n+1} = \frac{2^{-n}}{3n} + \left(1 - \frac{1}{3n}\right)I_n.$$ [5]

(ii) By considering the graph of $y = \frac{1}{1 + x^3}$, show that $I_1 < 1$. [2]

(iii) Deduce that $I_3 < \frac{53}{72}$. [3]

10 The curve C has equation

$$y = \frac{x^2 + 2x - 3}{(\lambda x + 1)(x + 4)},$$

where λ is a constant.

(i) Find the equations of the asymptotes of C for the case where $\lambda = 0$. [4]

(ii) Find the equations of the asymptotes of C for the case where λ is not equal to any of -1, 0, $\frac{1}{4}$, $\frac{1}{3}$. [3]

(iii) Sketch C for the case where $\lambda = -1$. Show, on your diagram, the equations of the asymptotes and the coordinates of the points of intersection of C with the coordinate axes. [4]

11 The line l_1 passes through the point A, whose position vector is $3\mathbf{i} - 5\mathbf{j} - 4\mathbf{k}$, and is parallel to the vector $3\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$. The line l_2 passes through the point B, whose position vector is $2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$, and is parallel to the vector $\mathbf{i} - \mathbf{j} - 4\mathbf{k}$. The point P on l_1 and the point Q on l_2 are such that PQ is perpendicular to both l_1 and l_2. The plane Π_1 contains PQ and l_1, and the plane Π_2 contains PQ and l_2.

(i) Find the length of PQ. [4]

(ii) Find a vector perpendicular to Π_1. [2]

(iii) Find the perpendicular distance from B to Π_1. [3]

(iv) Find the angle between Π_1 and Π_2. [3]

12 Answer only **one** of the following two alternatives.

EITHER

The variable y depends on x, and the variables x and t are related by $x = e^t$. Show that

$$x\frac{dy}{dx} = \frac{dy}{dt} \quad \text{and} \quad x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}. \qquad [5]$$

(i) Given that y satisfies the differential equation

$$4x^2\frac{d^2y}{dx^2} + 16x\frac{dy}{dx} + 25y = 50(\ln x) - 1,$$

find a differential equation involving only t and y. [2]

(ii) Show that the complementary function of the differential equation in t and y may be written in the form

$$Re^{-\frac{3}{2}t}\sin(2t + \phi),$$

where R and ϕ are arbitrary constants. [3]

(iii) Find a particular integral of the differential equation in t and y. [3]

(iv) Hence find the general solution of the differential equation in x and y. [1]

OR

The matrix **A** has λ as an eigenvalue with **e** as a corresponding eigenvector. Show that if **A** is non-singular then

(i) $\lambda \neq 0$, [2]

(ii) the matrix $\mathbf{A}^{-1}$ has λ^{-1} as an eigenvalue with **e** as a corresponding eigenvector. [2]

The matrices **A** and **B** are given by

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 2 \\ 0 & -2 & 4 \\ 0 & 0 & -3 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = (\mathbf{A} + 4\mathbf{I})^{-1}.$$

Find a non-singular matrix **P**, and a diagonal matrix **D**, such that $\mathbf{B} = \mathbf{PDP}^{-1}$. [10]

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