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**FURTHER MATHEMATICS**

**9231/12**

Paper 1

**October/November 2018**

MARK SCHEME

Maximum Mark: 100

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**Published**

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

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This document consists of **20** printed pages.

**PUBLISHED****Generic Marking Principles**

These general marking principles must be applied by all examiners when marking candidate answers. They should be applied alongside the specific content of the mark scheme or generic level descriptors for a question. Each question paper and mark scheme will also comply with these marking principles.

**GENERIC MARKING PRINCIPLE 1:**

Marks must be awarded in line with:

- the specific content of the mark scheme or the generic level descriptors for the question
- the specific skills defined in the mark scheme or in the generic level descriptors for the question
- the standard of response required by a candidate as exemplified by the standardisation scripts.

**GENERIC MARKING PRINCIPLE 2:**

Marks awarded are always **whole marks** (not half marks, or other fractions).

**GENERIC MARKING PRINCIPLE 3:**

Marks must be awarded **positively**:

- marks are awarded for correct/valid answers, as defined in the mark scheme. However, credit is given for valid answers which go beyond the scope of the syllabus and mark scheme, referring to your Team Leader as appropriate
- marks are awarded when candidates clearly demonstrate what they know and can do
- marks are not deducted for errors
- marks are not deducted for omissions
- answers should only be judged on the quality of spelling, punctuation and grammar when these features are specifically assessed by the question as indicated by the mark scheme. The meaning, however, should be unambiguous.

**GENERIC MARKING PRINCIPLE 4:**

Rules must be applied consistently e.g. in situations where candidates have not followed instructions or in the application of generic level descriptors.

**GENERIC MARKING PRINCIPLE 5:**

Marks should be awarded using the full range of marks defined in the mark scheme for the question (however; the use of the full mark range may be limited according to the quality of the candidate responses seen).

**GENERIC MARKING PRINCIPLE 6:**

Marks awarded are based solely on the requirements as defined in the mark scheme. Marks should not be awarded with grade thresholds or grade descriptors in mind.

**Abbreviations**

AEF/OE Any Equivalent Form (of answer is equally acceptable) / Or Equivalent

AG Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)

CAO Correct Answer Only (emphasising that no “follow through” from a previous error is allowed)

CWO Correct Working Only – often written by a ‘fortuitous’ answer

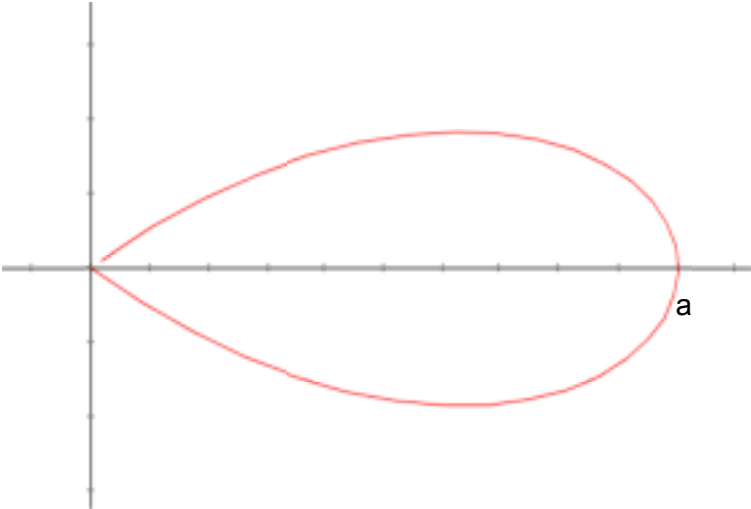
ISW Ignore Subsequent Working

SOI Seen or implied

SR Special Ruling (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)

| Question | Answer  | Marks     | Guidance  |
|----------|---|-----------|---|
| 1(i)     | $\alpha + \beta + \gamma = 5, \alpha\beta + \alpha\gamma + \beta\gamma = 13$  | <b>B1</b> | Sum of roots and $\alpha\beta + \alpha\gamma + \beta\gamma$ . SOI |
|          | $\alpha^2 + \beta^2 + \gamma^2 = 5^2 - 2(13)$   | <b>M1</b> | Uses $\sum \alpha^2 = (\sum \alpha)^2 - 2(\sum \alpha\beta)$      |
|          | $= -1$  | <b>A1</b> | www   |
|          |   | <b>3</b>  |   |
| 1(ii)    | $\alpha^3 + \beta^3 + \gamma^3 = 5(\alpha^2 + \beta^2 + \gamma^2) - 13(\alpha + \beta + \gamma) + 12$   | <b>M1</b> | Uses $\alpha^3 = 5\alpha^2 - 13\alpha + 4$ .                      |
|          | $= 5(-1) - 13(5) + 12 = -58$  | <b>A1</b> |   |
|          | Alt method: Use formula e.g. $\sum \alpha^3 = (\sum \alpha)(\sum \alpha^2 - \sum \alpha\beta) + 3\alpha\beta\gamma$<br>Or $(\sum \alpha)^3 - 3(\sum \alpha)(\sum \alpha\beta) + 3\alpha\beta\gamma$ |           |   |
|          |   | <b>2</b>  |   |

| Question | Answer   | Marks     | Guidance  |
|----------|--|-----------|---|
| 2(i)     | 2  | <b>B1</b> | Stated  |
|          |  | <b>1</b>  |   |
| 2(ii)    | Negative eigenvalue = -2   | <b>B1</b> | Stated  |
|          | $\mathbf{A} + 2\mathbf{I} = \begin{pmatrix} 4 & 3 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{pmatrix} \quad \left  \begin{array}{ccc} i & j & k \\ 4 & 3 & 1 \\ 0 & 0 & 1 \end{array} \right $ | <b>M1</b> | Uses vector product (or equations) to find corresponding eigenvector.                 |
|          | $\begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix}$   | <b>A1</b> | Accept any non-zero scalar multiple of $\begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix}$ . |
|          |  | <b>3</b>  |   |
| 2(iii)   | An eigenvalue of $A + A^6$ is $2 + 2^6 = 66$ , 62 or 2   | <b>B1</b> |   |
|          | Corresponding eigenvector is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ , $\begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} -6 \\ 1 \\ 3 \end{pmatrix}$ oe             | <b>B1</b> |   |
|          |  | <b>2</b>  |   |

| Question | Answer   | Marks     | Guidance  |
|----------|--|-----------|---|
| 3(i)     |  | <b>B1</b> | Just one loop, correct shape at extremities             |
|          |  | <b>B1</b> | Correct position including (a, 0) labelled or in table. |
|          |  | <b>2</b>  |   |

| Question | Answer   | Marks     | Guidance   |
|----------|--|-----------|--|
| 3(ii)    | $\frac{1}{2} \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} a^2 \cos^2 3\theta \, d\theta$  | <b>M1</b> | For using correct formula                        |
|          | $\frac{a^2}{4} \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} (\cos 6\theta + 1) \, d\theta$  | <b>M1</b> | Using double angle formula correctly             |
|          | $= \frac{a^2}{4} \left[ \frac{1}{6} \sin 6\theta + \theta \right]_{-\frac{\pi}{6}}^{\frac{\pi}{6}} = \frac{\pi a^2}{12}$                 | <b>A1</b> |  |
|          |  | <b>3</b>  |  |
| 3(iii)   | $r = a \cos \theta (4 \cos^2 \theta - 3) \Rightarrow r = a \left( \frac{x}{r} \right) \left( 4 \left( \frac{x}{r} \right)^2 - 3 \right)$ | <b>B1</b> | Uses $x = r \cos \theta$ and $x^2 + y^2 = r^2$ . |
|          | $\Rightarrow r^4 = ax(4x^2 - 3r^2) \Rightarrow (x^2 + y^2)^2 = ax(4x^2 - 3(x^2 + y^2))$  | <b>M1</b> | For eliminating $\theta$                         |
|          | $\Rightarrow (x^2 + y^2)^2 = ax(x^2 - 3y^2)$   | <b>A1</b> | Any equivalent cartesian form without fractions. |
|          |  | <b>3</b>  |  |

| Question | Answer  | Marks       | Guidance  |
|----------|---|-------------|---|
| 4(i)     | $m^2 + 2m + 1 = 0 \Rightarrow (m + 1)^2 = 0 \Rightarrow m = -1$                         | <b>M1</b>   | Forms and solves auxiliary equation.              |
|          | CF: $(A + Bt)e^{-t}$  | <b>A1</b>   | States CF.  |
|          | PI: $x = p \sin t + q \cos t$   | <b>M1</b>   | Uses correct form of PI and differentiates twice. |
|          | $\Rightarrow \dot{x} = p \cos t - q \sin t \Rightarrow \ddot{x} = -p \sin t - q \cos t$ | <b>A1</b>   |   |
|          | $-p \sin t - q \cos t + 2(p \cos t - q \sin t) + p \sin t + q \cos t = 4 \sin t$        | <b>M1</b>   | Compares coefficients and attempts to solve       |
|          | $2p = 0 \Rightarrow p = 0. -2q = 4 \Rightarrow q = -2.$                                 | <b>A1</b>   |   |
|          | <b>GS:</b> $x = (A + Bt)e^{-t} - 2 \cos t$  | <b>A1FT</b> | States general solution. FT on correct form only  |
|          | <b>7</b>  |             |   |
| 4(ii)    | $x \approx -2 \cos t$   | <b>B1FT</b> |   |
|          |   | <b>1</b>    |   |

| Question | Answer   | Marks     | Guidance                  |
|----------|--|-----------|---------------------------|
| 5(i)     | $\begin{pmatrix} 3 & 2 & 0 & 1 \\ 6 & 5 & -1 & 3 \\ 9 & 8 & -2 & 5 \\ -3 & -2 & 0 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 3 & 2 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ | <b>M1</b> | Attempt to row reduction. |
|          | $\rightarrow \begin{pmatrix} 3 & 2 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$  | <b>A1</b> | Two correct rows only     |
|          | $r(\mathbf{M}) = 4 - 2 = 2$  | <b>A1</b> | Obtains rank.             |
|          |  | <b>3</b>  |                           |

| Question | Answer  | Marks       | Guidance                                |
|----------|---|-------------|---|
| 5(ii)    | $\begin{aligned} 3x + 2y + t &= 0 \\ y - z + t &= 0 \end{aligned}$  | <b>M1</b>   | Solves homogeneous system of equations. |
|          | $\Rightarrow t = \mu, z = \lambda, y = \lambda - \mu, x = -\frac{2}{3}\lambda + \frac{1}{3}\mu$   | <b>M1</b>   | Using 2 parameters                      |
|          | A basis is $\left\{ \begin{pmatrix} -2 \\ 3 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ 0 \\ 3 \end{pmatrix} \right\}$ or $\left\{ \begin{pmatrix} -1 \\ 0 \\ 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \\ 2 \end{pmatrix} \right\}$ or equivalent | <b>A1</b>   | AEF                                     |
|          |   | <b>3</b>    |   |
| 5(iii)   | $\mathbf{M} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 8 \\ -2 \end{pmatrix} \text{ so a particular solution is } \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$   | <b>B1</b>   | Finds a particular solution.            |
|          | General solution: $(\mathbf{x} =) \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 3 \\ 3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -3 \\ 0 \\ 3 \end{pmatrix}$   | <b>M1</b>   | Using correct format                    |
|          |   | <b>A1FT</b> |   |
|          |   | <b>3</b>    |   |

| Question | Answer  | Marks       | Guidance   |
|----------|---|-------------|--|
| 6        | $y^{(1)} = e^x u^{(1)} + u e^x = e^x \left( \binom{1}{0} u + \binom{1}{1} u^{(1)} \right) \Rightarrow H_1$ is true  | <b>M1A1</b> | Shows base case using product rule   |
|          | Assume that<br>$H_k : y^{(k)} = e^x \left( \binom{k}{0} u + \binom{k}{1} u^{(1)} + \dots + \binom{k}{r} u^{(r)} + \dots + \binom{k}{k} u^{(k)} \right)$   | <b>B1</b>   | States inductive hypothesis.   |
|          | Then<br>$y^{(k+1)} = e^x \left( \binom{k}{0} u + \binom{k}{1} u^{(1)} + \dots + \binom{k}{r} u^{(r)} + \dots + \binom{k}{k} u^{(k)} \right) +$  | <b>M1</b>   | Differentiates using product rule  |
|          | $e^x \left( \binom{k}{0} u^{(1)} + \binom{k}{1} u^{(2)} + \dots + \binom{k}{r} u^{(r+1)} + \dots + \binom{k}{k} u^{(k+1)} \right)$<br>$= e^x \left( \binom{k}{0} u + \dots \left( \binom{k}{r} + \binom{k}{r-1} \right) u^r + \dots \binom{k}{k} u^{(k+1)} \right)$ | <b>M1A1</b> | Shows application of<br>$\binom{k}{r} + \binom{k}{r-1} = \binom{k+1}{r}$ . |
|          | $= e^x \left( \binom{k+1}{0} u + \dots \binom{k+1}{r} u^r + \dots \binom{k+1}{k+1} u^{(k+1)} \right)$   | <b>B1</b>   | Shows reasoning for first and last term correctly                          |
|          | So $H_k$ implies $H_{k+1}$ so, by induction, $H_n$ is true for all $n \geq 1$ .   | <b>A1</b>   | States conclusion.   |
|          |   | <b>8</b>    |  |

| Question | Answer   | Marks | Guidance   |
|----------|--|-------|--|
| 7(i)     | $\sum_{r=1}^N (3r+1)(3r+4) = 9\sum_{r=1}^N r^2 + 15\sum_{r=1}^N r + 4N$  | M1    | Expands  |
|          | $9\left(\frac{1}{6}N(N+1)(2N+1)\right) + 15\left(\frac{1}{2}N(N+1)\right) + 4N$  | M1    | Substitutes formulae for $\sum r$ and $\sum r^2$ . |
|          | $= N\left(\frac{9}{6}(2N^2 + 3N + 1) + \frac{15}{2}N + \frac{15}{2} + 4\right)$<br>$= N(3N^2 + 12N + 13)$                            | A1    | Shows simplification to the given answer (AG).     |
|          |  | 3     |  |
| 7(ii)    | $\frac{1}{(3r+1)(3r+4)} = \frac{1}{3}\left(\frac{1}{3r+1} - \frac{1}{3r+4}\right)$   | B1    | Finds partial fractions.                           |
|          | $T_N = \frac{1}{3}\left(\frac{1}{4} - \frac{1}{7} + \frac{1}{7} - \frac{1}{10} + \dots + \frac{1}{3(N-1)+1} - \frac{1}{3N+4}\right)$ | M1    | Expresses terms as differences.                    |
|          | $\frac{1}{3}\left(\frac{1}{4} - \frac{1}{3N+4}\right) = \frac{1}{12} - \frac{1}{3(3N+4)}$  | A1    | Cancels to given answer (AG).                      |
|          |  | 3     |  |
| 7(iii)   | $T_N = \frac{N}{4(3N+4)} \Rightarrow \frac{S_N}{T_N} = 4(3N+4)(3N^2 + 12N + 13)$   | M1    | Writes $\frac{S_N}{T_N}$ as a polynomial           |
|          | So $\frac{S_N}{T_N}$ is an integer because all terms are integers  | A1    | Justifies expression being integer                 |
|          |  | 2     |  |

| Question | Answer  | Marks     | Guidance   |
|----------|---|-----------|--|
| 7(iv)    | $\frac{S_N}{N^3 T_N} = \frac{4(3N+4)(3N^2+16N+9)}{N^3}$ | <b>M1</b> | Divides expression in (iii) by $N^3$ and takes limit |
|          | $\rightarrow 4(3)(3) = 36$                              | <b>A1</b> |  |
|          |   | <b>2</b>  |  |

| Question | Answer   | Marks       | Guidance                                      |
|----------|--|-------------|---|
| 8(i)     | $z + z^{-1} = 2 \cos \theta$   | <b>B1</b>   | Use of $z + z^{-1} = 2 \cos \theta$ .         |
|          | $(z + z^{-1})^6 = (z^6 + z^{-6}) + 6(z^4 + z^{-4}) + 15(z^2 + z^{-2}) + 20$                      | <b>M1A1</b> | Expands and groups.                           |
|          | $64 \cos^6 \theta = 2 \cos 6\theta + 12 \cos 4\theta + 30 \cos 2\theta + 20$                     | <b>M1A1</b> | Substitutes $z^n + z^{-n} = 2 \cos n\theta$ . |
|          | $\Rightarrow \cos^6 \theta = \frac{1}{32}(10 + 15 \cos 2\theta + 6 \cos 4\theta + \cos 6\theta)$ | <b>A1</b>   | (Allow $p=10, q=15, r=6, s=1$ .)              |
|          |  | <b>6</b>    |   |

| Question | Answer   | Marks | Guidance                                  |
|----------|--|-------|---|
| 8(ii)    | $\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \cos^6 \frac{x}{2} dx = \frac{1}{32} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} 10 + 15 \cos x + 6 \cos 2x + \cos 3x dx$               | M1    | Applies part (i)                          |
|          |  | M1    | Integrates correctly (3/4 terms correct). |
|          | $\frac{1}{32} \left[ 10x + 15 \sin x + 3 \sin 2x + \frac{1}{3} \sin 3x \right]_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}$   | M1    | Inserts limits and evaluates.             |
|          | $= \frac{1}{32} \left\{ \left( 5\pi + 15 + 0 - \frac{1}{3} \right) - \left( -5\pi - 15 + 0 + \frac{1}{3} \right) \right\} = \frac{1}{16} \left( 5\pi + \frac{44}{3} \right)$ | A1    |   |
|          |  | 4     |   |

| Question | Answer  | Marks | Guidance                  |
|----------|---|-------|---------------------------|
| 9(i)     | $y = 5 - \frac{4}{x^2 + x + 1}$                                       | M1    | Alt method: Finding limit |
|          | As $x \rightarrow \pm\infty$ , $y \rightarrow 5 \therefore y = 5$ CAO | A1    |                           |
|          |   | 2     |                           |

| Question | Answer   | Marks     | Guidance                                |
|----------|--|-----------|---|
| 9(ii)    | $yx^2 + yx + y = 5x^2 + 5x + 1$<br>$\Rightarrow (y-5)x^2 + (y-5)x + (y-1) = 0$ | <b>B1</b> | Forms quadratic equation in $x$ .       |
|          | For real $x$ , $(y-5)^2 - 4(y-5)(y-1) \geq 0$ (condone $>$ )                   | <b>M1</b> | Uses discriminant                       |
|          | $\Rightarrow (y-5)(3y+1) \leq 0$   | <b>M1</b> | Factorising                             |
|          | $\Rightarrow -\frac{1}{3} \leq y < 5$ , because $y = 5$ is an asymptote (www)  | <b>A1</b> | Explaining strict upper inequality (AG) |
|          |  | <b>4</b>  |   |
| 9(iii)   | $y' = 0 \Rightarrow (x^2 + x + 1)(10x + 5) - (5x^2 + 5x + 1)(2x + 1) = 0$      | <b>M1</b> | Differentiates and equates to 0.        |
|          | $\Rightarrow 4(2x + 1) = 0 \Rightarrow x = -\frac{1}{2}, y = -\frac{1}{3}$     | <b>A1</b> |   |
|          |  | <b>2</b>  |   |

| Question | Answer | Marks       | Guidance   |
|----------|--------|-------------|--|
| 9(iv)    |        | <b>B1FT</b> | Positive $y$ -intercept at $(0,1)$ , FT dep on minimum point from <b>(iii)</b> . |
|          |        | <b>B1</b>   | Correct asymptote and completely correct graph.                                  |
|          |        | <b>2</b>    |  |

| Question | Answer   | Marks     | Guidance                           |
|----------|--|-----------|------------------------------------|
| 10(i)    | $\overrightarrow{AB} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}, \overrightarrow{CD} = \begin{pmatrix} 1 \\ 1.5 \\ 1 \end{pmatrix} = \frac{1}{2} \overrightarrow{AB}$ | <b>B1</b> | Or shows if parallel, then $m=3/2$ |
|          |  | <b>1</b>  |                                    |

| Question | Answer   | Marks         | Guidance  |
|----------|--|---------------|---|
| 10(ii)   | $\overline{AB} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}, \overline{CD} = \begin{pmatrix} 1 \\ m \\ 1 \end{pmatrix}$ and $\overline{AC} = \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}$ or $AD = \begin{pmatrix} -1 \\ m-1 \\ 1 \end{pmatrix}$ or $BC = \begin{pmatrix} -4 \\ -4 \\ -2 \end{pmatrix}$ | <b>B1</b>     |   |
|          | $\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 2 \\ 1 & m & 1 \end{vmatrix} = \begin{pmatrix} 3-2m \\ 0 \\ 2m-3 \end{pmatrix}$ (so parallel to $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ )   | <b>M1A1</b>   | Finds common perpendicular using cross product. |
|          | $\frac{ AC \cdot \mathbf{n} }{ \mathbf{n} } = \frac{ -2(3-2m)+0+0 }{\sqrt{(3-2m)^2 + (2m-3)^2}}$ o.e.  | <b>M1</b>     | Uses formula for shortest distance.             |
|          | $= \frac{2}{\sqrt{2}} = \sqrt{2}$  | <b>A1</b>     |   |
|          |  | <b>5</b>      |   |
| 10(iii)  | $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 2 \\ -2 & -1 & 0 \end{vmatrix} = \begin{pmatrix} 2 \\ -4 \\ 4 \end{pmatrix} \sim \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ o.e.   | <b>M1A1</b>   | Finds normal to plane $ABC$ (AEF).              |
|          | $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 2 \\ -1 & 1 & 1 \end{vmatrix} = \begin{pmatrix} 1 \\ -4 \\ 5 \end{pmatrix}$ o.e.  | <b>A1</b>     | Finds normal to $ABD$ (AEF).                    |
|          | $\cos \theta = \frac{1+8+10}{\sqrt{1^2+2^2+2^2} \sqrt{1^2+4^2+5^2}} \left( = \frac{19}{\sqrt{378}} \right)$  | <b>M1A1FT</b> | Uses formula for angle between two lines.       |
|          | $\Rightarrow \theta = 12.2^\circ$  | <b>A1</b>     | CAO.  |
|          |  | <b>6</b>      |   |

| Question | Answer  | Marks      | Guidance   |
|----------|---|------------|--|
| 11E(i)   | $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{12t^{\frac{1}{2}}}{18-2t} = \frac{6t^{\frac{1}{2}}}{9-t}$ | <b>B1</b>  | AEF.   |
|          | $\frac{d^2y}{dx^2} = \frac{d}{dt} \left( \frac{dy}{dx} \right) \times \frac{dt}{dx}$                                  | <b>M1</b>  | Uses chain rule again to find second derivative.       |
|          | $= \frac{3t^{-\frac{1}{2}}(9-t) + 6t^{\frac{1}{2}}}{(9-t)^2(18-2t)}$  | <b>dM1</b> | Uses quotient (or product rule)                        |
|          | $= \frac{3t^{-\frac{1}{2}}(9-t+2t)}{2(9-t)^3} = \frac{3(9+t)}{2t^{\frac{1}{2}}(9-t)^3}$                               | <b>A1</b>  | AG.  |
|          |   | <b>4</b>   |  |
| 11E(ii)  | $\frac{1}{56} \int_0^{56} \frac{d^2y}{dx^2} dx$   | <b>B1</b>  | Uses correct formula for mean value                    |
|          | $= \frac{1}{56} \left[ \frac{6t^{\frac{1}{2}}}{9-t} \right]_{t=0}^{t=4}$  | <b>M1</b>  | Finding limits correctly                               |
|          |   | <b>M1</b>  | Using expression                                       |
|          | $= \frac{1}{56} \left( \frac{6\sqrt{4}}{9-4} \right) = \frac{3}{70}$  | <b>A1</b>  | Inserts correct values of $t$ and obtains answer (AG.) |
|          |   | <b>4</b>   |  |

| Question | Answer  | Marks       | Guidance   |
|----------|---|-------------|--|
| 11E(iii) | $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (18-2t)^2 + 144t = 4(t+9)^2$                     | <b>M1A1</b> | Simplifies $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$ . |
|          | $2\pi \int_0^4 \left(8t^{\frac{3}{2}}\right)(2(t+9)) dt = 32\pi \int_0^4 t^{\frac{5}{2}} + 9t^{\frac{3}{2}} dt$ | <b>M1A1</b> | Uses $2\pi \int y \frac{ds}{dt} dt$ .                                      |
|          | $= 32\pi \left[ \frac{2}{7}t^{\frac{7}{2}} + \frac{18}{5}t^{\frac{5}{2}} \right]_0^4$                           | <b>M1</b>   | Integrates term by term.   |
|          | $= \frac{2^{11} \times 83}{35} \pi = \frac{169984}{35} \pi = 15300$   | <b>A1</b>   | Accept exact answer or decimal rounding to 15300.                          |
|          |   | <b>6</b>    |  |
| 11O(i)   | $I_n = \left[ x(x^2-1)^n \right]_1^{\sqrt{2}} - 2n \int_1^{\sqrt{2}} x^2(x^2-1)^{n-1} dx$                       | <b>M1A1</b> | Integrates by parts.   |
|          | $= \sqrt{2} - 2n \int_1^{\sqrt{2}} (x^2-1+1)(x^2-1)^{n-1} dx$   | <b>M1</b>   | Uses $x^2 = x^2 - 1 + 1$ .   |
|          | $= \sqrt{2} - 2nI_n - 2nI_{n-1}$  | <b>A1</b>   |  |
|          | $\Rightarrow (2n+1)I_n = \sqrt{2} - 2nI_{n-1}$  | <b>A1</b>   | AG.  |
|          |   | <b>5</b>    |  |

| Question | Answer   | Marks       | Guidance   |
|----------|--|-------------|--|
| 11O(ii)  | $\frac{dx}{d\theta} = \tan \theta \sec \theta$   | <b>M1A1</b> | Differentiates $\sec \theta$ .                         |
|          | $\sec \theta = \sqrt{2} \Rightarrow \theta = \frac{\pi}{4} \quad \sec \theta = 1 \Rightarrow \theta = 0$   | <b>B1</b>   | Changes limits.  |
|          | $I_n = \int_0^{\frac{\pi}{4}} (\sec^2 \theta - 1)^n \tan \theta \sec \theta d\theta = \int_0^{\frac{\pi}{4}} \tan^{2n+1} \theta \sec \theta d\theta$ | <b>B1</b>   | Uses $\sec^2 \theta - 1 = \tan^2 \theta$ . (AG.)       |
|          |  | <b>4</b>    |  |
| 11O(iii) | $\int_0^{\frac{\pi}{4}} \frac{\sin^7 \theta}{\cos^8 \theta} d\theta = I_3$   | <b>B1</b>   | Deduces that integral is $I_3$ .                       |
|          | $I_0 = \sqrt{2} - 1 \quad (\text{or } I_1 = \frac{2 - \sqrt{2}}{3})$   | <b>B1</b>   | Calculates $I_0$ or $I_1$                              |
|          | $3I_1 = \sqrt{2} - 2I_0 \Rightarrow I_1 = \frac{2 - \sqrt{2}}{3}$  | <b>M1A1</b> | Uses reduction formula to find $I_1$ or $I_2$<br>..... |
|          | $I_2 = \frac{\sqrt{2}}{5} - \frac{4}{5}I_1 = \frac{7\sqrt{2} - 8}{15}$   |             | Finds $I_2$ .  |
|          | $I_3 = \frac{\sqrt{2}}{7} - \frac{6}{7}I_2 = \frac{16 - 9\sqrt{2}}{35}$  | <b>A1</b>   | Finds $I_3$ .  |
|          |  | <b>5</b>    |  |