Cambridge International A Level

Cambridge Assessment International Education
Cambridge International Advanced Level

FURTHER MATHEMATICS **9231/21**

Paper 2 **October/November 2018**

MARK SCHEME

Maximum Mark: 100

Published

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge International will not enter into discussions about these mark schemes.

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Generic Marking Principles

These general marking principles must be applied by all examiners when marking candidate answers. They should be applied alongside the specific content of the mark scheme or generic level descriptors for a question. Each question paper and mark scheme will also comply with these marking principles.

GENERIC MARKING PRINCIPLE 1:

Marks must be awarded in line with:

- the specific content of the mark scheme or the generic level descriptors for the question
- the specific skills defined in the mark scheme or in the generic level descriptors for the question
- the standard of response required by a candidate as exemplified by the standardisation scripts.

GENERIC MARKING PRINCIPLE 2:

Marks awarded are always **whole marks** (not half marks, or other fractions).

GENERIC MARKING PRINCIPLE 3:

Marks must be awarded **positively**:

- marks are awarded for correct/valid answers, as defined in the mark scheme. However, credit is given for valid answers which go beyond the scope of the syllabus and mark scheme, referring to your Team Leader as appropriate
- marks are awarded when candidates clearly demonstrate what they know and can do
- marks are not deducted for errors
- marks are not deducted for omissions
- answers should only be judged on the quality of spelling, punctuation and grammar when these features are specifically assessed by the question as indicated by the mark scheme. The meaning, however, should be unambiguous.

GENERIC MARKING PRINCIPLE 4:

Rules must be applied consistently e.g. in situations where candidates have not followed instructions or in the application of generic level descriptors.

GENERIC MARKING PRINCIPLE 5: Marks should be awarded using the full range of marks defined in the mark scheme for the question (however; the use of the full mark range may be limited according to the quality of the candidate responses seen).
GENERIC MARKING PRINCIPLE 6: Marks awarded are based solely on the requirements as defined in the mark scheme. Marks should not be awarded with grade thresholds or grade descriptors in mind.

Abbreviations

AEF/OE Any Equivalent Form (of answer is equally acceptable) / Or Equivalent

AG Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)

CAO Correct Answer Only (emphasising that no "follow through" from a previous error is allowed)

CWO Correct Working Only – often written by a 'fortuitous' answer

ISW Ignore Subsequent Working

SOI Seen or implied

SR Special Ruling (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)

Question	Answer	Marks	Guidance
1	$\omega = 2\pi/T = 4$	**B1**	Find ω from period T (may be implied)
	$v = \omega\sqrt{(a^2 - x^2)} = 4\sqrt{(3^2 - 1^2)} = 8\sqrt{2}$ *or* 11.3 [m s^{-1}]	**M1A1**	Find speed v when $BP = 2$
		3	

Question	Answer	Marks	Guidance
2(i)	$5mv_A + 2mv_B = 5mu - 4mu = mu$ (AEF)	**M1**	Use momentum (allow m omitted)
	$v_B - v_A = e(u + 2u) = 3eu$	**M1**	Use Newton's law (M0 if LHS signs inconsistent)
	$v_A = (u/7)(1 - 6e)$	**A1**	Combine to find/verify speeds of A and B after colln.
	$v_B = (u/7)(1 + 15e)$ AG	**A1**	(ignore signs)
		4	

Question	Answer	Marks	Guidance
2(ii)	$(u/7)(1 - 6e) = -\frac{1}{2}u$, $e = \frac{3}{4}$ *or* 0.75	**M1A1**	Combine to find e from $v_A = -\frac{1}{2}u$ (M0 if dirn. of motion not reversed)
		2	
2(iii)	$KE_A = \frac{1}{2} \times 5m\ \{u^2 - (\frac{1}{2}u)^2\}$ *and* $[= (15/8)mu^2]$ $KE_B = \frac{1}{2} \times 2m\ \{(2u)^2 - (7u/4)^2\}$ $[= (15/16)mu^2]$	**M1A1**	Find loss of KE for A and B
	$KE_A / KE_B = (15/8) / (15/16) = 2:1$ *or* 2/1 *or* 2	**A1**	Combine to find ratio
		3	

Question	Answer	Marks	Guidance
3(i)	$I_{AB} = \frac{1}{3} M (3a)^2 + M (3a)^2$ *or* $(4/3) M (3a)^2$ $[= 12 Ma^2]$	**B1**	Find or state MI of rod AB about axis l
	$I_{\text{disc}} = \frac{1}{2}\, 2\, M a^2 + 2M x^2$ $[= Ma^2 + 2M x^2]$	**M1A1**	Find MI of disc about axis l
	$I = 13\, M a^2 + 2M x^2$	**A1**	Find MI of object about axis l
		4	
3(ii)	$\frac{1}{2} I \omega^2 = Mg \times 3a \cos\theta + 2Mg\, x \cos\theta$	**M1A1**	Find ω^2 or angular speed ω at angle θ by energy, with $\cos\theta = 3/5$
	$\omega^2 = 6(3a + 2x)g / 5(13\, a^2 + 2x^2)$	**A1**	
	$6a(3a + 2x) = 2(13\, a^2 + 2x^2)$ $x^2 - 3ax + 2a^2 = 0$, $x = a$ *or* $2a$	**M1A1**	Equate ω^2 to $2g/5a$ and solve quadratic for x
		5	

Question	Answer	Marks	Guidance
4(i)	$T \times 3a \sin 2\theta = W \times 2a \cos \theta + ½ W \times 4a \cos \theta$	**M1A1**	Take moments for rod about A
	$6a\ T \sin \theta \cos \theta = 4a\ W \times a \cos \theta$	**M1**	Verify tension T
	$T = 2W / 3 \sin \theta = 2W / 3(8/17) \quad = 17W / 12$ AG	**A1**	using $\sin \theta = 8/17$, $\cos \theta = 15/17$
		4	
4(ii)	*EITHER:* $X = T \cos \theta \quad = (5/4)\ W$ *or* $1.25W$	**B1**	Find horizontal component X of force at A
	$Y = W + ½ W - T \sin \theta \quad = (5/6)\ W$ *or* $0.833W$	**B1**	Find vertical component Y of force at A
	$F = \sqrt{(X^2 + Y^2)} = (5\sqrt{13} / 12)\ W$ *or* $1.50W$	**B1**	Find magnitude of F
	Upward force at angle to horizontal of $\tan^{-1} Y/X = \tan^{-1} 2/3 = 33.7°$ *or* 0.588 radians	**M1A1**	Find direction of F (AEF; A0 if direction unclear)
	OR: $R_P \quad = 1.5\ W \sin \theta + T \cos 2\theta$ $= (305/12 \times 17)\ W$ *or* $1.45W$	**(B1**	Find component R_P parallel to AB of force at A
	$R_N \quad = 1.5\ W \cos \theta - T \cos 2\theta$ $= (5/34)\ W$ *or* $0.147W$	**B1**	Find component R_N normal to AB of force at A
	$F = \sqrt{(R_P{}^2 + R_N{}^2)} = (5\sqrt{13} / 12)\ W$ *or* $1.50W$	**B1**	Find magnitude of F
	Upward force at angle to AB of $\tan^{-1} R_N / R_P = \tan^{-1} 6/61$ $= 5.6°$ *or* 0.098 radians	**M1A1)**	Find direction of F (AEF; A0 if direction unclear)
		5	

Question	Answer	Marks	Guidance
4(iii)	$T = \lambda\,(CD - 2a)/2a$	**M1**	Find modulus λ using Hooke's Law
	$CD = 3a$ so $\lambda = 2T = 17W / 6$ *or* $2.83W$	**A1**	
		2	

Question	Answer	Marks	Guidance
5(i)	$\lambda\,(AE - 1.25) / 1.25 =$	**M1A1**	Verify AE by equating equilibrium tensions
	$0.6\lambda\,(2.6 - AE - 1.0) / 1.0$ (AEF)	**A1**	**SC**: vertical motion can earn M1 only
	$AE - 1.25 = 1.2 - 0.75\,AE$, $AE = 1.4$ **AG**	**A1**	
		4	
5(ii)	$[\pm]\, m\, \mathrm{d}^2x/\mathrm{d}t^2 = -\lambda\,(0.15 + x) / 1.25 + 0.6\lambda\,(0.2 - x) / 1.0$ *or* $+\lambda\,(0.15 - x) / 1.25 - 0.6\lambda\,(0.2 + x) / 1.0$	**M1A2**	Apply Newton's law at $1.4 + x$ or $1.4 - x$ from A (lose A1 for one incorrect tension term)
	$\mathrm{d}^2x/\mathrm{d}t^2 = -(1.4\,\lambda / 0.4)\,x = -3.5\,\lambda x$	**M1A1**	Simplify to give SHM eqn. in standard form (A0 if no minus sign, or dirn. of acceln. is undefined) **SC**: B2 if result stated without derivation (max 2/5)
		5	
5(iii)	$\omega = 2\pi / (\pi/7)$ [= 14] $= \sqrt{(3.5\,\lambda)}$	**M1A1**	Find eqn. for λ using $T = 2\pi/\omega$ with **FT** on ω from SHM eqn.
	$\lambda = 14^2 / 3.5 = 56$	**A1**	Solve for λ
		3	

Question	Answer	Marks	Guidance
6(i)	$F(x) = \int f(x)\,dx = (1/80)\,(2x^{3/2} - 16x^{1/2})$ [+ c]	**M1**	Find or state distribution function $F(x)$ for $4 \leqslant x \leqslant 16$ using $F(4) = 0$ or $F(16) = 1$ to find c if necessary
	$= (1/80)\,(2x^{3/2} - 16x^{1/2} + 16)$ *or* $(1/40)\,(x^{3/2} - 8x^{1/2} + 8)$ *or* $x^{3/2}/40 - x^{1/2}/5 + 1/5$ (AEF)	**A1**	State $F(x)$ for other values of x
	$F(x) = 0$ $(x < 4)$, $F(x) = 1$ $(x > 16)$	**A1**	
		3	
6(ii)	*EITHER:* $G(y)$ [$= P(Y < y) = P(\sqrt{X} < y) = P(X < y^2)$] $= F(y^2) = (1/40)\,(y^3 - 8y + 8)$ (AEF)	**M1A1**	Find or state $G(y)$ for $2 \leqslant y \leqslant 4$ from $Y = \sqrt{X}$ (allow < or $\leqslant$ throughout; **FT** on constant term)
	OR: Use $x = y^2$ to find $f(x) = (1/80)\,(3y - 8/y)$ *and* $dx/dy = 2y$	**(M1A1)**	Find $f(x)$ *and* dx/dy for use in $g(y) = f(x) \times \lvert dx/dy \rvert$
	$g(y)$ [$= G'(y)$] $= (1/40)\,(3y^2 - 8)$ *or* $(3/40)\,y^2 - 1/5$ [for $2 \leqslant y \leqslant 4$, $g(y) = 0$ otherwise]	**A1**	Find $g(y)$ in simplified form for $2 \leqslant y \leqslant 4$
		3	

Question	Answer	Marks	Guidance
7(i)	f(t) = (1/500) exp (– t/500) [0 otherwise] (AEF)	**B1**	State pdf of T for $t \geqslant 0$
		1	
7(ii)	F(t) = 1 – exp (– t/500)	**M1**	Find or imply F(t)
	P(T > 750) = 1 – F(750) = exp (– 750/500) = 0.223	**M1A1**	Find P(T > 750) (M0 for F(750))
		3	
7(iii)	1 – exp (– m/500) = ½ *or* exp (– m/500) = ½	**M1**	Find median value m of T from F(m) = ½
	m = 500 ln 2 = 347 [hours]	**M1A1**	
		3	

Question	Answer	Marks	Guidance
8(i)	$s_A^2 = (531\,000 - 5120^2/50) / 49$ *and* [= 6712/49] $s_B^2 = (375\,135 - 3760^2/40) / 39$ [= 21695/39]	**M1**	Estimate both population variances
	$s_A^2 = 136.98$ *and* $s_B^2 = 556.28$ (to 3 s.f. throughout)	**A1**	(allow biased here: 134.24 *and* 542.4)
	EITHER: $s^2 = s_A^2/50 + s_B^2/40 = 16.65$ *or* 4.08^2	**M1A1**	*EITHER:* Estimate combined variance
	$z_{0.975} = 1.96$	***B1**	State or use correct tabular z (or t) value
	$z = (\bar{y} - \bar{x}) / s = (102.4 - 94) / s = 2.06$ $z >$ tabular value [accept H_1]	**M1A1**	Calculate value of z (or $-z$) (or can compare $\bar{y} - \bar{x} = 8.4$ with 8.0)
	so $\mu_A \neq \mu_B$ *or* salaries do differ (AEF)	**B1**	Correct conclusion (**FT** on z, dep *B1)
	OR: Assume equal [population] variances $s^2 = (49\ s_A^2 + 39\ s_B^2) / 88$ *or* $(531\,000 - 5120^2/50 + 375\,135 - 3760^2/40) / 88$	**(B1**	*OR:* State assumption about variances (may be in part (ii)) Find pooled estimate of common variance (M1 A1 for s_A^2 and s_B^2 may be implied here)
	$= 28407/88$ *or* 322.8 *or* 17.97^2	**B1**	
	$z_{0.975} = 1.96$	***B1**	State or use correct tabular z (or t) value
	$z = 8.4 / s\sqrt{(1/50 + 1/40)} = 2.20$	**M1A1**	Calculate value of z (or $-z$) (or can compare $\bar{y} - \bar{x} = 8.4$ with 7.47)
	$z >$ tabular value [accept H_1] so $\mu_A \neq \mu_B$ *or* salaries do differ (AEF)	**B1)**	Correct conclusion (**FT** on z, dep *B1)
		8	

Question	Answer	Marks	Guidance
8(ii)	No assumption [of normality] needed since large samples *or* due to central limit theorem	**B1**	Valid comment on required assumptions (AEF) (assumption of equal variances may earn B1 above)
		1	

Question	Answer	Marks	Guidance
9(i)	$\bar{x}$ = 13.88/8 = 1.735 (allow 1.73 or 1.74 for this B1)	**B1**	Find sample mean
	s^2 = (24.1182 – 13.88^2 /8) / 7 [= 13/2500 *or* 0.0052 *or* 0.07211^2]	**M1**	Estimate population variance (allow biased here: 0.00455 *or* 0.06745^2)
	H_0: μ = 1.7, H_1: μ > 1.7 (AEF)	**B1**	State hypotheses (B0 for $\bar{x}$ ...)
	t = ($\bar{x}$ – 1.7) / ($s/\sqrt{8}$) = 1.37	**M1A1**	Find value of t
	$t_{7, 0.95}$ = 1.89[5]	**B1**	State or use correct tabular t-value (or can compare $\bar{x}$ with 1.7 + 0.048 = 1.75)
	[Accept H_0:] Mean ht. not greater than 1.7 m (AEF)	**B1**	Consistent conclusion (**FT** on both t-values)
		7	
9(ii)	13.88/8 ± t $\sqrt{(s^2/8)}$	**M1**	Find confidence interval
	$t_{7, 0.975}$ = 2.36[5]	**A1**	State or use correct tabular value of t
	1.73[5] ± 0.06 *or* [1.67, 1.80]	**A1**	Evaluate confidence interval (either form)
		3	

Question	Answer	Marks	Guidance
10(i)	$\bar{x} = (\bar{y} - 1.1664) / 0.4604 = 11.02$ *or* $\Sigma x = 110.2$	**M1**	Find $\bar{x}$ or Σx using eqn. of regression line of y on x
	$S_{xx} = 1419.98 - 110.2^2/10 = 205.576$ $S_{yy} = 439.68 - 62.4^2/10 = 50.304$ (AEF)	**M1**	Find S_{xx} and S_{yy} (may be earned in part **(ii)**)
	EITHER: $b = S_{xy} / S_{yy} = (S_{xy} / S_{xx}) \times (S_{xx} / S_{yy})$ $= 0.4604 \times 205.576 / 50.304 = 1.88$[15]	**M1A1**	Find gradient b to 3 s.f. in $x - \bar{x} = b\,(y - \bar{y})$
	OR: $S_{xy} = 0.4604 \times S_{xx} = 94.647$ $b = S_{xy} / S_{yy} = 94.647 / 50.304 = 1.88$[15]	**(M1A1)**	
	$(x - 11.02) = b\,(y - 6.24)$, $x = 1.88y - 0.721$	**M1A1**	and hence eqn. of regression line of x on y
		6	
10(ii)	$r = S_{xy} / \sqrt{(S_{xx} S_{yy})} = 94.647 / \sqrt{(205.576 \times 50.304)}$ *or* $\sqrt{(0.4604 \times b)} = \sqrt{(0.4604 \times 1.8815)}$ (AEF)	**M1**	Find correlation coefficient r
	$= 0.931$	***A1**	
		2	
10(iii)	H_0: $\rho = 0$, H_1: $\rho > 0$	**B1**	State both hypotheses (B0 for r …)
	EITHER: $r_{10,\,5\%} = 0.549$	***B1**	State or use correct tabular one-tail r-value
	Reject H_0 if $\lvert r \rvert >$ tab. r-value (AEF)	**M1**	State or imply valid method for conclusion
	OR: $t_r = r\sqrt{((n-2) / (1 - r^2))} = 7.21$, $t_{8,0.95} = 1.86$	**(*B1**	(Rarely seen)
	Reject H_0 if $\lvert t_r \rvert >$ tab. t-value (AEF)	**M1)**	
	Evidence of positive correlation (AEF)	**A1**	Correct conclusion (dep *A1, *B1)
		4	

Question	Answer	Marks	Guidance
11A(i)	$\frac{1}{2}mv_C^2 = \frac{1}{2}mu^2 + mga\,(1 - \cos\alpha)$ [$v_C^2 = u^2 + (2/5)\,ag$]	**M1A1**	Find speed v_C at C by conservation of energy (A0 if no m)
	$R_A = mg\cos\alpha + mu^2/a$	**B1**	Find reaction R_A at A by using $F = ma$
	$R_C = mg + mv_C^2/a$	**B1**	Find reaction R_C at C by using $F = ma$
	$8\,\{ag + u^2 + (2/5)\,ag\} = 9\,\{(4/5)\,ag + u^2\}$	**M1**	Verify u^2 using $R_C / R_A = 9/8$
	$u^2 = (8 + 16/5 - 36/5)\,ag = 4ag$ AG	**A1**	
		6	
11A(ii)	$\frac{1}{2}mv_B^2 = \frac{1}{2}mu^2 - 2mga\cos\alpha$ *or* $\frac{1}{2}mv_C^2 - mga\,(1 + \cos\alpha)$ [$v_C^2 = (22/5)\,ag$]	**M1**	Find speed v_B at B by conservation of energy (A0 if no m)
	$v_B^2 = (4 - 16/5)\,ag$ *or* $(22/5 - 18/5)\,ag = (4/5)\,ag$	***A1**	
	$\geqslant 0$ so P reaches height of B (AEF; dep *A1)	**A1**	(or valid argument based on R_A)
	$R_A = m\,v_B^2/a - mg\cos\alpha = 4mg/5 - 4mg/5 = 0$	**M1**A1**	Find reaction R_B at B by using $F = ma$
	$\geqslant 0$ so P still [just] in contact at B (AEF; dep **A1)	**A1**	
		6	
11B(i)	$\bar{x} = (1/200)\,\Sigma\, x\, f(x) = 30\,064 / 200 = 150.3$[2]	**M1A1**	Find sample mean to 4 s.f. using mid-interval values (B1 for 29 964 / 200 = 149.8[2] or 30 164 / 200 = 150.8[2])
		2	

Question	Answer	Marks	Guidance
11B(ii)	P(151 ⩽ X < 152) = P((151 – 150)/1.2 ⩽ Z < (152 – 150)/1.2) = P(0.833 ⩽ Z < 1.667) = 0.9522 – 0.7975 [= 0.1547] (to 4 s.f.)	**M1A1**	Find P(151 ⩽ X < 152)
	E_{151} = 190.44 – 159.5 *or* 200 × 0.1547 = 30.94 AG	**A1**	and hence expected frequency for 151 ⩽ x < 152
		3	
11B(iii)	H_0: [Normal] distribution fits data (AEF)	**B1**	State (at least) null hypothesis in full
	O_i: 3 23 52 69 36 17 E_i: 9.56 30.94 59.50 59.50 30.94 9.56	**M1**	Combine values consistent with all exp. values ⩾ 5
	X^2 = 4.501 + 2.038 + 0.945 + 1.517 + 0.828 + 5.790	**M1**	Find value of X^2 from Σ $(E_i - O_i)^2 / E_i$ [or Σ $O_i^2/E_i - n$]
	= 15.6 (to 3 s.f.)	**A1**	
	No. n of cells: 8 7 6 5 $\chi^2_{n-1,\,0.95}$: 14.07 12.59 11.07 9.488	**B1**	State or use consistent tabular value $\chi^2_{n-1,\,0.95}$ (to 3 s.f.) [**FT** on number, n, of cells used to find X^2]
	Accept H_1 if X^2 > tabular value (AEF)	**M1**	State or imply valid method for conclusion
	15.6 [± 0.1] > 11.1 so distn. doesn't fit [data] (AEF)	**A1**	Conclusion (requires both values correct)
		7	